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CONSTRUCTING PRODUCT FIBRATIONS BY MEANS OF A GENERALIZATION OF A THEOREM OF GANEA

PAUL SELICK

Abstract. A theorem of Ganea shows that for the principal homotopy fibration $\Omega B \to F \to E$ induced from a fibration $F \to E \to B$, there is a product decomposition $\Omega(E/F) \approx \Omega B \times \Omega(F * \Omega B)$. We will determine the conditions for a fibration $X \to Y \to Z$ to yield a product decomposition $\Omega(Z/Y) \approx X \times \Omega(X * Y)$ and generalize it to pushouts. Using this approach we recover some decompositions originally proved by very computational methods. The results are then applied to produce, after localization at an odd prime p, homotopy decompositions for $\Omega J_k\left(S^{2n}\right)$ for some k which include the cases $k = p^t$. The factors of $\Omega J_{p^t}\left(S^{2n}\right)$ consist of the homotopy fibre of the attaching map $S^{2np^t-1} \to J_{p^t-1}\left(S^{2n}\right)$ for $J_{p^t}\left(S^{2n}\right)$ and combinations of spaces occurring in the Snaith stable decomposition of $\Omega^2 S^{2n+1}$.

0. Introduction

Over the past fifteen years a number of product decompositions up to homotopy $X \approx Y \times Z$ for simply connected H-spaces X have been constructed by guessing the appropriate Y and Z (after examination of $H_*(X; \mathbb{Z}/p\mathbb{Z})$ and its possible coalgebra and Steenrod module decompositions) and then constructing maps $Y \to X$, $Z \to X$ which induce appropriate maps on homology such that the composite $Y \times Z \to X \times X \to X$ induces an isomorphism on homology. Although this method has had a number of successes it is intrinsically difficult to apply because it requires complete knowledge of the homology of X, good intuition to select appropriate Y and Z, and ad hoc methods to construct suitable maps $Y \to X$, $Z \to X$. The need is felt for more powerful methods which require less calculation and do not demand that the user guess the proposed decomposition in advance. This paper is concerned with a method based on a well known theorem of Ganea.

Let $F \to E \to B$ be a fibration. Ganea's theorem [G] states that the homotopy fibre of the induced map $E/F \to B$ is homotopy equivalent to $F * \Omega B$. In fact we get a map of homotopy fibrations

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \| \\ F * \Omega B & \longrightarrow & E/F & \longrightarrow & B. \end{array}$$

Since the canonical map $M \to M * N$ is always null homotopic, the left map $F \to F * \Omega B$ and the map $\Omega B \to F * \Omega B$ induced from the bottom homotopy

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fibration are null homotopic. A consequence of the latter map being null is the following corollary.

Corollary 0.1. *Let $F \to E \to B$ be a homotopy fibration. Then*

$$\Omega(E/F) \approx \Omega B \times \Omega(F * \Omega B).$$

Observe that B does not appear directly in this formula but only in the combination ΩB. Thus regarding the induced homotopy fibration $\Omega B \to F \to E$ as the model case, it is natural to ask under what conditions the existence of a homotopy fibration $X \to Y \to Z$ implies a product decomposition

$$(*) \qquad \Omega(Z/Y) \approx X \times \Omega(X * Y)$$

where Z/Y denotes the homotopy cofibre of the map $Y \to Z$.

It is easy to construct counterexamples to the existence of such a decomposition for an arbitrary homotopy fibration $X \to Y \to Z$ so some hypotheses will be needed. For example, such a product decomposition exhibits X as a retract of a loop space and so there is no chance if X is not an H-space. Nevertheless there are cases where $(*)$ holds even though unlike the situation to which Ganea's theorem applies, the initial homotopy fibration is not principal. Consider for example the degree p map $p : S^{2n+1} \to S^{2n+1}$ for an odd prime p. Let $S^{2n+1}\{p\}$ denote the homotopy fibre of p and let $P^{2n+2}(p) = S^{2n+1} \cup_p e^{2n+2}$ denote the Moore space which is the homotopy cofibre of p. Then $(*)$ predicts that

$$\Omega P^{2n+2}(p) \approx S^{2n+1}\{p\} \times \Omega(S^{2n+1}\{p\} * S^{2n+1}).$$

Since $M * N \approx \Sigma M \wedge N$ and $\Sigma S^{2n+1}\{p\} \approx \bigvee_{k=0}^{\infty} P^{2n+2+2nk}(p)$ this becomes

$$\Omega P^{2n+2}(p) \approx S^{2n+1}\{p\} \times \Omega \left(\bigvee_{k=0}^{\infty} P^{4n+3+2nk}(p) \right).$$

The latter formula is in fact a theorem of Cohen, Moore, and Neisendorfer [CMN1] so formula $(*)$ holds in this case. Notice however that our starting fibration $S^{2n+1}\{p\} \to S^{2n+2} \xrightarrow{p} S^{2n+1}$ is not principal, since $S^{2n+1}\{p\}$ is not even a loop space.

The first goal of this paper is to give the conditions which imply that formula $(*)$ holds for a given fibration $X \to Y \to Z$ and to consider a more general situation in which the cofibre Z/Y is replaced by a pushout. Theorem 1.6 and its CW reformulation Corollary 1.7 present if and only if conditions in terms of a fibre homotopy equivalence between certain pullbacks. Section 2 deals with certain conditions which are shown to be sufficient to guarantee the hypotheses of the preceding theorems. For example, Corollary 2.2 states that $(*)$ holds if Y is an H-space such that the map $Y \to Z$ extends to a homotopy action of Y on Z. Section 3 applies the results of sections 1 and 2 to construct some homotopy decompositions of $\Omega J_k\left(S^{2n}\right)$, the loop space on the k th James filtration on S^{2n}. As shown in [S4], the mod p homology of these spaces has exponential growth for most k, although Moore's conjecture (cf. [S3]) predicts that they should all have a homotopy exponent at the prime p. Among the cases for which a decomposition is obtained in section 3 are those where k equals p^t for some t. The factors which appear in the decomposition of $\Omega J_{p^t}\left(S^{2n}\right)$ consist of the homotopy fibre $F_{2t}(n)$ of the attaching map $S^{2np^t-1} \to J_{p^t-1}\left(S^{2n}\right)$ for $J_{p^t}\left(S^{2n}\right)$ together with spaces formed by suspensions and smash products of the spaces $D_{j,2}(S^{2n-1})$ for $j \leq p^t$ which appear in the Snaith stable decomposition of $\Omega^2 S^{2n+1}$. The question of whether these spaces related to $D_{j,2}(S^{2n-1})$ have mod p homotopy exponents is thus seen to be related to the question of whether the spaces $J_{p^t}\left(S^{2n}\right)$ have homotopy exponents.

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1. Fibrations over pushouts

We begin by reviewing the work of Dold and Thom [DT] on quasi-fibrations.

Notation. Given pointed spaces M and N we let $\pi' : M \times N \to M$ and $\pi'' : M \times N \to N$ denote the projections and let $i' : M \to M \times N$ and $i'' : N \to M \times N$ denote the inclusions $i'(m) = (m, *)$, $i''(n) = (*, n)$. We use $\approx$ for a homotopy equivalence and $\approx_w$ for a weak homotopy equivalence. We will sometimes use the Milnor-Moore convention where the name of a space is used to denote the identity map of that space.

Definition 1.1 (Dold). A map $p : E \to B$ is a *quasi-fibration* if for all $b \in B$ the induced map $p_\# : \pi_q(E, p^{-1}b) \to \pi_q(B, b)$ is an isomorphism for all q. If $p : E \to B$ is a quasi-fibration then $p^{-1}b$ is weak homotopy equivalent to the homotopy fibre of p for all $b \in B$.

Given a map $p : E \to B$ and a subset $A \subset B$ we will sometimes write E_A for $p^{-1}A$ and p_A for $p|_A$. The subset A will be called *distinguished* if $p_A : E_A \to A$ is a quasi-fibration.

Theorem 1.2 (Dold-Thom). *Suppose there exists an open cover $\{U_\alpha\}_{\alpha \in J}$ of B by distinguished sets U_α such that for each pair α, $\beta \in J$ and for all $b \in U_\alpha \cap U_\beta$ there exists distinguished open V such that $b \in V \subset U_\alpha \cap U_\beta$. Then p is a quasi-fibration.* $\square$

From Theorem 1.2 we can deduce the following.

Theorem 1.3. *Let $\nu : D \to X$ and $p : E \to Y$ be fibrations. Let $A \hookrightarrow X$ and $A \hookrightarrow Y$ be inclusions such that A becomes a closed subset of both X and Y. Let $\phi : D_A \xrightarrow{\approx} E_A$ be a weak homotopy equivalence such that*

$$\begin{tikzcd} D_A \arrow[rr, "\phi"] \arrow[dr, "\nu_A"'] & & E_A \arrow[dl, "p_A"] \\ & C & \end{tikzcd}$$

commutes. Suppose that A has open neighbourhoods U_X in X and U_Y in Y with strong deformation retracts $U_X \to A$ and $U_Y \to A$. Then $\pi : Z \to B$ is a quasi-fibration, where the pushouts

$$\begin{tikzcd} D_A \arrow[r, "\phi"] \arrow[d, hook] & E_A \arrow[r, hook] & E \arrow[d] \\ D \arrow[rr, "\phi"] & & Z \end{tikzcd}, \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & Y \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

define Z and B and $\pi : Z \to B$ is the induced map.

Proof. It is not hard to check that the conditions of Theorem 1.2 are satisfied for the open cover $\{X - A,\ Y - A,\ U_X \cap U_Y\}$ of B. $\square$

By analogy with Lusternik-Schnirrelmann category we say that a space has *numerable category* if it has a numerable open cover by subsets with null homotopic inclusion maps. If A has numerable category (for example, a CW-complex) and ϕ is a homotopy equivalence then ν_A is fibre homotopy equivalent to p_A.

As in Theorem 1.3, let $i : A \hookrightarrow X$, $j : A \hookrightarrow Y$ be such that $i(A)$ and $j(A)$ are closed in X and Y respectively and have open neighbourhoods with strong deformation retracts back to the image of A. As before let B be the pushout of i and j. Let G_1 and G_2 be the homotopy fibres of i and j respectively. It is well known (see [CMN2]) that the homotopy pullback of $G_1 \to A$ and $G_2 \to A$ is the homotopy fibre both of the composite $G_1 \to A \to Y$ and of the composite $G_2 \to A \to X$. That is, we have a diagram of homotopy fibrations

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & Y \arrow[d, equal] \\ G_2 \arrow[r] \arrow[d] & A \arrow[r, "j"] \arrow[d, "i"] & Y \\ X \arrow[r, equal] & X & \end{tikzcd}$$

where F is the common homotopy fibre. Our goal is to give conditions under which we have a homotopy fibration $F \to G_1 * G_2 \to B$ with null homotopic $F \to G_1 * G_2$, and consequently a product decomposition $\Omega B \approx F \times \Omega(G_1 * G_2)$. Notice that in the special case where $Y \approx *$ we have $G_1 \approx F$ and $G_2 \approx A$ so this would yield $(*)$ for the homotopy fibration $F \to A \to X$. Let P, Q, f, g, p, and q be defined from the homotopy pullbacks

$$\begin{tikzcd} P \arrow[rr] \arrow[d, "f"] & & A \arrow[d, "i"] \\ G_2 \arrow[r] & A \arrow[r, "i"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P \arrow[rr] \arrow[d, "g"] & & A \arrow[d, "j"] \\ G_1 \arrow[r] & A \arrow[r, "j"] & Y. \end{tikzcd}$$

Thus the homotopy fibre of $P \to G_2$ is G_1, the same as that of i. Furthermore using the definition of pullback, this homotopy fibration $G_1 \to P \to G_2$ has a cross-section $G_2 \to P$ induced by the maps $G_2 \to A$ and $1_{G_2} : G_2 \to G_2$. Similarly we have a homotopy fibration with cross-section $G_2 \to Q \to G_1$. It is not unreasonable to suppose that in many cases we will have an equivalence $\phi : P \approx G_1 \times G_2 \approx Q$ such that

$$\begin{tikzcd} P \arrow[rr, "\phi"] \arrow[dr, "p"] & & Q \arrow[dl, "q"] \\ & A & \end{tikzcd}$$

commutes. Some conditions which imply this will be considered in the next section. If we have such a ϕ, then we can apply Theorem 1.3 to the maps obtained by turning i and j into fibrations to obtain a quasi-fibration $F \to Z \to B$. However it is possible that the resulting fibration might be uninteresting; for example, Z might be $F \times B$. Further hypotheses are needed to obtain the desired $G_1 * G_2$ as Z.

Example 1.4. Let A be an H-space and let i and j each be the inclusion $A \hookrightarrow CA$ of A into the cone on A. Then $F \approx G_1 \approx G_2 \approx A$, $B = \Sigma A$, and up to homotopy equivalence, P and Q are each given by the pullbacks

$$\begin{array}{ccc} A\times A & \xrightarrow{\pi'} & A \\ \big\downarrow{\scriptstyle \pi''} & & \big\downarrow \\ A & \longrightarrow & *. \end{array}$$

If we let $\phi = 1_{A\times A}$ then Z becomes the homotopy pushout

$$\begin{tikzcd} A\times A \arrow[r, equal] \arrow[d, "\pi''"'] & A\times A \arrow[r, "\pi''"] & A \arrow[d] \\ A \arrow[rr] & & Z \end{tikzcd}$$

which is homotopy equivalent to $A \times \Sigma A$ which in this case is equivalent to $F \times B$. If we had instead chosen ϕ to be the shearing map $sh_2 : A\times A \xrightarrow{\approx} A\times A$ given by $sh(x, y) = (x, xy)$ then

$$\begin{tikzcd} A\times A \arrow[rr, "sh_2"] \arrow[dr, "\pi'"'] & & A\times A \arrow[dl, "\pi'"] \\ & A & \end{tikzcd}$$

would still have commuted but (as we shall see) we would have obtained $Z \approx A * A$. □

Lemma 1.5. *Let $G_1 \xrightarrow{j_1} P \xrightarrow{f} G_2$ and $G_2 \xrightarrow{j_2} Q \xrightarrow{g} G_1$ be homotopy fibrations and let $\phi : P \to Q$ be a weak equivalence. Then the following are equivalent:*

(i) *$P \xrightarrow{g\phi\times f} G_1 \times G_2$ is a weak equivalence,*

(ii) *the composite $G_1 \xrightarrow{j_1} P \xrightarrow{\phi} Q \xrightarrow{g} G_1$ is a weak equivalence,*

(iii) *the composite $G_2 \xrightarrow{j_2} Q \xrightarrow{\phi^{-1}} P \xrightarrow{f} G_2$ is a weak equivalence.*

Proof. The homotopy fibration diagram

$$\begin{array}{ccc} G_1 & \overset{\alpha}{\dashrightarrow} & G_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle i'} \\ P & \xrightarrow{g\phi\times f} & G_1\times G_2 \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \pi''} \\ G_2 & = & G_2 \end{array}$$

shows that $g\phi \times f$ is a weak equivalence if and only if α is a weak equivalence. However $\alpha = \pi' i' \alpha = \pi' \circ (g\phi \times f) \circ j_1 = g\phi j_1$. Thus (i) is equivalent to (ii). By symmetry $g\phi \times f$ is a weak equivalence if and only if $f\phi^{-1} j_2$ is a weak equivalence. □

Theorem 1.6. *Let $i : A \hookrightarrow X$, $j : A \hookrightarrow Y$ be inclusions such that A becomes a closed subset of both X and Y. Suppose that A has open neighbourhoods U_X in X and U_Y in Y with strong deformation retracts $U_X \to A$ and $U_Y \to A$. Let B be the homotopy pushout of i and j, let G_1 and G_2 be the homotopy fibres of i and j and let F be the homotopy pullback of $G_1 \to A$ and $G_2 \to A$. Let P be the homotopy pullback of i and the composite $G_2 \to A \xrightarrow{i} X$ and let Q be the homotopy pullback of j and the composite $G_1 \to A \xrightarrow{j} Y$. Suppose there exists a weak equivalence*

$\phi : P \to Q$ such that

$$\begin{tikzcd} P \arrow[rr, "\phi"] \arrow[dr] & & Q \arrow[dl] \\ & A & \end{tikzcd}$$

commutes. Suppose also that the composite $G_1 \to P \xrightarrow{\phi} Q \to G_1$ is a weak equivalence. Then there exists a quasi-fibration $F \to Z \to B$ with Z weak homotopy equivalent to $G_1 * G_2$ such that there are homotopy commutative diagrams

$$\begin{array}{ccccc} F & \longrightarrow & G_2 & \longrightarrow & X \\ \downarrow{\scriptstyle \approx_w} & & \downarrow & & \downarrow \\ F & \longrightarrow & Z & \longrightarrow & B \end{array} \quad \textit{and} \quad \begin{array}{ccccc} F & \longrightarrow & G_1 & \longrightarrow & Y \\ \downarrow{\scriptstyle \approx_w} & & \downarrow & & \downarrow \\ F & \longrightarrow & Z & \longrightarrow & B. \end{array}$$

where the composites $G_2 \to Z \xrightarrow{\approx_w} G_1 * G_2$ and $G_1 \to Z \xrightarrow{\approx_w} G_1 * G_2$ are the canonical (null homotopic) maps. Consequently there is a weak homotopy equivalence

$$\Omega B \approx F \times \Omega(G_1 * G_2).$$

Conversely if there exists a quasi-fibration $F \to Z \to B$ making the above diagrams commute with Z weak equivalent to $G_1 * G_2$ then there exists a weak equivalence ϕ with the two properties above.

Proof. Suppose first that there exists a weak equivalence $\phi : P \to Q$ with the stated properties. Applying Theorem 1.3 to the maps obtained by turning $G_2 \to X$ and $G_1 \to Y$ into fibrations yields a quasi-fibration $F \to Z \to B$ and the homotopy commutative diagrams

$$\begin{array}{ccccc} F & \longrightarrow & G_2 & \longrightarrow & X \\ \downarrow{\scriptstyle \approx_w} & & \downarrow & & \downarrow \\ F & \longrightarrow & Z & \longrightarrow & B \end{array} \quad \text{and} \quad \begin{array}{ccccc} F & \longrightarrow & G_1 & \longrightarrow & Y \\ \downarrow{\scriptstyle \approx_w} & & \downarrow & & \downarrow \\ F & \longrightarrow & Z & \longrightarrow & B. \end{array}$$

We must show that $Z \approx_w G_1 * G_2$ and that the maps $G_1 \to Z \xrightarrow{\approx_w} G_1 * G_2$ and $G_2 \to Z \xrightarrow{\approx_w} G_1 * G_2$ are null. By definition, Z is the homotopy pushout

$$\begin{array}{ccccc} P & \xrightarrow{\phi} & Q & \xrightarrow{g} & G_1 \\ \downarrow{\scriptstyle f} & & & & \downarrow \\ G_2 & & \longrightarrow & & Z. \end{array}$$

By Lemma 1.5, $g\phi \times f : P \to G_1 \times G_2$ is a weak homotopy equivalence. It is standard that

$$\begin{array}{ccc} M \times N & \xrightarrow{\pi'} & M \\ \downarrow{\scriptstyle \pi''} & & \downarrow \\ N & \longrightarrow & M * N \end{array}$$

is a homotopy pushout. Thus in the commutative cube

$$\begin{tikzcd}
P \arrow[rr, "\phi"] \arrow[dr, "g\phi\times f"] \arrow[dd, "f"'] & & \phi \quad Q \arrow[r, "g"] & G_1 \arrow[dr, "1_{G_1}"] \arrow[dd] & \\
 & G_1 \times G_2 \arrow[rrr, "\pi'"] \arrow[dd, "\pi''"] & & & G_1 \arrow[dd] \\
G_2 \arrow[rrr] \arrow[dr, "1_{G_2}"'] & & & Z \arrow[dr] & \\
 & G_2 \arrow[rrr] & & & G_1 * G_2
\end{tikzcd}$$

the front and back faces are homotopy pushouts and the maps from back face to the front face in the upper left, lower left, and upper right are weak homotopy equivalences. It follows that the induced map $Z \to G_1 * G_2$ in the lower right is also a weak equivalence and the diagram shows that the composites $G_1 \to Z \xrightarrow{\approx_w} G_1 * G_2$ and $G_2 \to Z \xrightarrow{\approx_w} G_1 * G_2$ are the canonical ones as desired.

Suppose now conversely that $F \to Z \to B$ is a quasi-fibration such that the given diagrams commute. The diagrams

$$\begin{array}{ccccc} F & \longrightarrow & P & \longrightarrow & A \\ \| & & \downarrow & & \downarrow \\ F & \longrightarrow & G_2 & \longrightarrow & X \\ \downarrow{\scriptstyle\approx_w} & & \downarrow & & \downarrow \\ F & \longrightarrow & Z & \longrightarrow & B \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} F & \longrightarrow & Q & \longrightarrow & A \\ \| & & \downarrow & & \downarrow \\ F & \longrightarrow & G_1 & \longrightarrow & Y \\ \downarrow{\scriptstyle\approx_w} & & \downarrow & & \downarrow \\ F & \longrightarrow & Z & \longrightarrow & B \end{array}$$

show that P is weak homotopy equivalent to the homotopy pullback of $Z \to B$ and $A \hookrightarrow X \to B$ and similarly Q is weak homotopy equivalent to the homotopy pullback of $Z \to B$ and $A \hookrightarrow Y \to B$. However the composites $A \hookrightarrow X \to B$ and $A \hookrightarrow Y \to B$ are equal. Therefore we have an weak equivalence $\phi : P \to Q$ such that

$$\begin{array}{ccc} P & \xrightarrow{\phi} & Q \\ & \searrow \quad \swarrow & \\ & A & \end{array}$$

commutes. In the above cube we know that the maps from the back face to the front face in the lower left, upper right, and lower right corner are weak equivalences. Thus the map $P \xrightarrow{g\phi\times f} G_1 \times G_2$ in the upper left corner is also an equivalence and so $G_1 \to P \xrightarrow{\phi} Q \to G_1$ is an equivalence by Lemma 1.5. $\square$

If $i : A \hookrightarrow X$ and $j : A \hookrightarrow Y$ are cellular maps of CW-complexes then neighbourhoods U_X and U_Y satisfying the conditions of Theorem 1.3 always exist. Since any map of CW-complexes can be replaced up to homotopy by a cellular inclusion we obtain the following corollary.

Corollary 1.7. *Let $i : A \to X$ and $j : A \to Y$ be maps between spaces having the homotopy type of CW-complexes. Let B be the homotopy pushout of i and j, let G_1 and G_2 be the homotopy fibres of i and j, and let F be the homotopy pullback of $G_1 \to A$ and $G_2 \to A$. Let P be the homotopy pullback of i and the composite*

$G_2 \to A \xrightarrow{i} X$ and let Q be the homotopy pullback of j and the composite $G_1 \to A \xrightarrow{j} Y$. Suppose that there is a homotopy equivalence $\phi : P \to Q$ such that

$$\begin{tikzcd} P \arrow[rr, "\phi"] \arrow[dr, "p"'] & & Q \arrow[dl, "q"] \\ & A & \end{tikzcd}$$

homotopy commutes and $G_1 \to P \xrightarrow{\phi} Q \to G_1$ is a homotopy equivalence. Then there is a quasi-fibration $F \to Z \to B$ with $Z \approx G_1 * G_2$ and homotopy commutative diagrams

$$\begin{array}{ccccc} F & \longrightarrow & G_2 & \longrightarrow & X \\ \big\downarrow{\scriptstyle\approx} & & \big\downarrow{\scriptstyle *} & & \big\downarrow \\ F & \longrightarrow & Z & \longrightarrow & B \end{array} \quad \text{and} \quad \begin{array}{ccccc} F & \longrightarrow & G_1 & \longrightarrow & Y \\ \big\downarrow{\scriptstyle\approx} & & \big\downarrow{\scriptstyle *} & & \big\downarrow \\ F & \longrightarrow & Z & \longrightarrow & B. \end{array}$$

Consequently $\Omega B \approx F \times \Omega(G_1 * G_2)$. □

Assuming still that A and X have the homotopy type of CW-complexes, we conclude this section by interpreting our hypotheses in the special case where $Y \approx *$. In this case we have $G_1 \approx F$, $G_2 \approx A$, P given by the homotopy pullback

$$\begin{array}{ccc} P & \xrightarrow{f} & A \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle i} \\ A & \xrightarrow{i} & X \end{array}$$

of i with itself, and $Q = F \times A$ with $g = \pi' : Q \to F$ and $q = \pi'' : Q \to A$. Since $q = \pi'' : Q \to A$ is a trivial fibration, the existence of ϕ is equivalent to the requirement that the homotopy pullback of i with itself, $p : P \to A$, be fibre homotopically trivial. Suppose now that we have such a fibre homotopy trivialization $\phi : P \to F \times A$. We can define an "action" $\mu : F \times A \to A$ of F on A by setting $\mu = f\phi^{-1}$. From the definitions the diagram

$$\begin{array}{ccc} F \times A & \xrightarrow{\mu} & A \\ \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle i} \\ A & \xrightarrow{i} & X \end{array}$$

is homotopy commutative with the restriction $\mu|_{F\times *}$ homotopic to the map $F \to A$ in the fibration $F \to A \to X$. The additional condition required to conclude that $(*)$ holds is that the restriction $\mu|_{*\times A} \to A$ be an equivalence.

Returning to the original case studied by Ganea where the fibration $F \to A \xrightarrow{i} X$ is induced from a fibration $A \xrightarrow{i} X \to BF$, we have homotopy pullback squares

$$\begin{array}{ccccc} P & \xrightarrow{p} & A & \longrightarrow & PBF \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle i} & & \big\downarrow \\ A & \xrightarrow{i} & X & \longrightarrow & BF. \end{array}$$

In this case the homotopy pullback of i with itself can be regarded as the pullback of $PBF \to B$ and the null homotopic map $A \xrightarrow{i} X \to BF$, and thus is fibre homotopically trivial. We can choose the trivialization so that $\mu : F \times A \to A$ becomes the standard action of the fibre of a principal fibration on the total space. For this action, the restriction $\mu|_{*\times A} \to A$ is homotopic to the identity map on A and so, of course, our hypotheses are satisfied in this case.

2. Some conditions sufficient to imply the decomposition

In this section we will consider a special case of the situation discussed in section 1 and show that the hypotheses of Corollary 1.7 are satisfied in this special case. Throughout this section every space will be assumed to have the homotopy type of a simply connected CW-complex.

Theorem 2.1. *Let $k : S \to X$ where S is an H-space. Suppose that $k \vee 1_X : S \vee X \to X$ extends to a homotopy action $\mu : S \times X \to X$ of S on X. That is, suppose there exists μ such that $\mu|_{S \vee X} \simeq k \vee 1_X$ and*

$$\begin{tikzcd} S \times S \arrow[r, "m"] \arrow[d, "S \times k"] & S \arrow[d, "k"] \\ S \times X \arrow[r, "\mu"] & X \end{tikzcd}$$

is homotopy commutative. Then for any space Y and any map $\gamma : Y \to X$ there exists a quasi-fibration $F \to R * S \to B$ where F is the homotopy fibre of $S \xrightarrow{k} X$, R is the homotopy pullback

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d, "k"] \\ Y \arrow[r, "\gamma"] & X \end{tikzcd},$$

and B is the homotopy pushout

$$\begin{tikzcd} S \times Y \arrow[r, "S \times \gamma"] \arrow[d, "\pi''"] & S \times X \arrow[r, "\mu"] & X \arrow[d] \\ Y \arrow[rr] & & B. \end{tikzcd}$$

Furthermore there are homotopy commutative diagrams

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & S \arrow[r, "k"] \arrow[d, "*"] & X \arrow[d] \\ F \arrow[r] & R * S \arrow[r] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[r] \arrow[d, equal] & R \arrow[r] \arrow[d, "*"] & Y \arrow[d] \\ F \arrow[r] & R * S \arrow[r] & B. \end{tikzcd}$$

In particular, $F \to R * S$ is null and so $\Omega B \approx F \times \Omega(R * S)$.

Proof. Using the homotopy extension property we may replace μ by a homotopic map such that $\mu|_{S \vee X} = k \vee 1_X$. Let A be $S \times Y$, let $i = \mu \circ (S \times \gamma) : A \to X$, and let $j = \pi'' : A \to Y$. We will show that the hypotheses of Corollary 1.7 are satisfied in this situation.

Examining the definitions we have

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d, "h"] & Y \arrow[d, equal] \\ G_2 = S \arrow[r, "i'"] \arrow[d, "k"] & A = S \times Y \arrow[r, "j = \pi''"] \arrow[d, "i = \mu \circ S \times \gamma"] & Y \\ X \arrow[r, equal] & X & \end{tikzcd}$$

and homotopy pullbacks

$$\begin{array}{ccc} P & \xrightarrow{p} & A = S \times Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i=\mu\circ S\times\gamma} \\ G_2 = S & \xrightarrow{k} & X \end{array} \quad , \qquad \begin{array}{ccc} Q = S \times G_1 & \xrightarrow{q=S\times\pi''h} & S \times Y \\ \downarrow{\scriptstyle g=\pi} & & \downarrow{\scriptstyle \pi''} \\ G_1 & \xrightarrow{\pi''h} & Y. \end{array}$$

The homotopy pullback defining P shows that the homotopy fibre of f is G_1 and yields an induced map $\tilde{h} : G_1 \to P$ such that $p\tilde{h} = h$.

Let $sh : S \times X \to S \times X$, $sh_1 : S \times S \to S \times S$, and $sh_2 : S \times S \to S \times S$ denote the shearing maps given by $sh(s, x) = \big(s, \mu(s, x)\big)$, $sh_1(s, t) = (st, t)$, and $sh_2(s, t) = (s, st)$. These are all homotopy equivalences. We sometimes write $m : S \times S \to S$ for the multiplication on S.

Since $\mu = \pi'' \circ sh$, we can expand the homotopy pullback square defining P to the composition of homotopy pullback squares

$$\begin{array}{ccc} P & \xrightarrow{p} & S \times Y \\ \downarrow & & \downarrow{\scriptstyle S\times\gamma} \\ S \times S & \xrightarrow{S\times k} & S \times X \\ \downarrow{\scriptstyle sh_2} & & \downarrow{\scriptstyle sh} \\ S \times S & \xrightarrow{S\times k} & S \times X \\ \downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi''} \\ S & \xrightarrow{k} & X. \end{array}$$

Since the pullback in the top square is clearly $S \times R$ this gives an equivalence $\psi : S \times R \to P$. We wish to construct a homotopy equivalence $R \approx G_1$ to be used to produce $\phi : P \approx Q$. Let $\alpha : R \to S$ and $\beta : R \to Y$ be the maps appearing in the homotopy pullback defining R and let G denote the homotopy fibre of the composite $S \times R \xrightarrow{S\times\alpha} S \times S \xrightarrow{m} S$. Notice that $f\psi = \pi'' \circ sh_2 \circ (S \times \alpha) = m \circ (S \times \alpha)$.

$$\begin{array}{ccccccc} S & \xleftarrow{\tau} & \tilde{S} & \longleftarrow & G & \xrightarrow{\sigma} & G_1 \\ \downarrow{\scriptstyle i''} & & \downarrow{\scriptstyle \tilde{i}} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \tilde{h}} \\ S \times S & \xleftarrow{sh_1} & S \times S & \xleftarrow{S\times\alpha} & S \times R & \xrightarrow{\psi} & P \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m\circ(S\times\alpha)} & & \downarrow{\scriptstyle f} \\ S & = & S & = & S & = & S \end{array}$$

defines the maps τ, σ, $\tilde{i}$, and ϵ, shows that τ and σ are homotopy equivalences, and shows that

$$\begin{array}{ccc} G & \longrightarrow & \tilde{S} \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \tilde{i}} \\ S \times R & \xrightarrow{S\times\alpha} & S \times S \end{array}$$

is a homotopy pullback square. In the composite homotopy pullback

$$\begin{array}{ccccc} G & \xrightarrow{\epsilon} & S \times Y & \xrightarrow{\pi''} & R \\ \downarrow & & \downarrow{\scriptstyle S\times\alpha} & & \downarrow{\scriptstyle \alpha} \\ \tilde{S} & \xrightarrow{\tilde{i}} & S \times S & \xrightarrow{\pi''} & S \end{array}$$

$\pi''\tilde{i} = \pi'' \circ sh_1 \circ \tilde{i} = \pi'' i'' \tau = \tau$ which is a homotopy equivalence. Therefore the composite $\pi'' \circ \epsilon : G \to R$ is also an equivalence. Let $\Phi : R \to G_1$ be the composition of homotopy equivalences $\sigma(\pi''\epsilon)^{-1}$ and define the equivalence $\phi : P \to Q$ to be the composite $P \xrightarrow{\psi^{-1}} S \times R \xrightarrow{S\times\Phi} S \times G_1 = Q$. Then

$$\begin{aligned}
q\phi &= (S \times \pi'' h) \circ \left(S \times \sigma(\pi''\epsilon)^{-1}\right) \circ \psi^{-1} \\
&= \left(S \times \pi'' h\sigma(\pi''\epsilon)^{-1}\right) \circ \psi^{-1} \\
&\simeq \left(S \times \pi'' p\tilde{h}\sigma(\pi''\epsilon)^{-1}\right) \circ \psi^{-1} \\
&= \left(S \times \pi'' p\psi\epsilon(\pi''\epsilon)^{-1}\right) \circ \psi^{-1} \\
&\simeq \left(S \times (\pi''(S\times\beta)\epsilon(\pi''\epsilon)^{-1})\right) \circ \psi^{-1} \\
&= \left(S \times \beta\pi''\epsilon(\pi''\epsilon)^{-1}\right) \circ \psi^{-1} \\
&\simeq (S \times \beta) \circ \psi^{-1} \\
&= p
\end{aligned}$$

as desired. Also the composite $G_2 = S \to Q = S \times G_2 \xrightarrow{\phi^{-1}} P \to G_2$ is given by $f\phi^{-1}i' = f\psi \circ (S\times\Phi)^{-1} \circ i' \simeq m \circ (S \times \alpha) \circ (S \times \Phi^{-1}) \circ i' = mi' = 1_S$. By Lemma 1.5 this is equivalent to showing the other condition in the hypotheses of Corollary 1.7. Thus the hypotheses of Corollary 1.7 are satisfied, and applying this together with our identifications $G_2 = S$ and $\Phi : R \approx G_1$ yields the theorem. □

Corollary 2.2. *Let $k : S \to X$ where S is an H-space and let F be the homotopy fibre of k. Suppose that $k \vee 1_X : S \vee X \to X$ extends to a homotopy action $\mu : S \times X \to X$ of S on X. Then there is a homotopy fibration $F \xrightarrow{*} F * S \to X/S$ and so $\Omega(X/S) \approx F \times \Omega(F * S)$.* □

Apply Theorem 2.1 with $Y = *$.

Corollary 2.3. *Let S and X be H-spaces, let $k : S \to X$ be an H-map and let F be the homotopy fibre of k. Then there is a homotopy fibration $F \xrightarrow{*} F * S \to X/S$ and so $\Omega(X/S) \approx F \times \Omega(F * S)$.* □

This last corollary yields Cohen, Moore, and Neisendorfer's formula for $\Omega P^{2n+2}(p)$, as observed in the introduction. The special case where $X = *$ yields the well known Hopf-like homotopy fibration $S \to S * S \to \Sigma S$ for an H-space S.

3. Application: some homotopy decompositions of $\Omega J_k\left(S^{2n}\right)$

Let p be an odd prime. Throughout this section we will assume that all spaces and maps have been localized at p and will use $H_*(\ )$ to denote homology with $\mathbb{Z}/p\mathbb{Z}$ coefficients. Let $J_k\left(S^{2n}\right)$ denote the kth filtration of the James construction on S^{2n}. As shown in [S4], $H_*\left(\Omega J_k\left(S^{2n}\right)\right)$ has exponential growth except when k has the form $qp^t - 1$ for some q and t. In [S4] it was shown how to get a decomposition of $\Omega J_k\left(S^{2n}\right)$ into atomic factors when $k < p^2 - p$ by means of the computational method outlined in the introduction to this paper. In this section we shall apply the techniques of sections 1 and 2 to obtain a homotopy decomposition of $\Omega J_k\left(S^{2n}\right)$ when k has the form $p^t + q$ with $q < p^t$. For $k < 2p$ the results duplicate those in [S4] although they are obtained here with much less effort.

James [J] has shown that $J(X) \approx \Omega\Sigma X$ and we shall sometimes abuse notation by identifying these spaces. Let $H_r : \Omega S^{2n+1} \to \Omega S^{2nr+1}$ denote the rth James-Hopf invariant map. It is well known that the homotopy fibre of H_{p^t} is $J_{p^t-1}\left(S^{2n}\right)$.

(See [T].) As in [S2] for all $t \geq 0$ we define $F_{2t}(n)$ to be the homotopy pullback

$$\begin{array}{ccc} F_{2t}(n) & \longrightarrow & \Omega S^{2n+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle \Omega H_{p^t}} \\ S^{2np^t-1} & \xrightarrow{E^2} & \Omega S^{2np^t+1} \end{array} .$$

As the pullback of H-maps E^2 and ΩH_{p^t} we know that $F_{2t}(n)$ is an H-space and the induced maps are H-maps. From [S2] and [CLM],

$$H_*\big(F_{2t}(n)\big) = \bigotimes_{j=0}^{t} \Lambda[v_{2np^j-1}] \otimes \bigotimes_{j=0}^{t} \mathbb{Z}/p\mathbb{Z}[u_{2np^j-2}]$$

with $\beta v_{2np^j-1} = u_{2np^j-2}$ and $\mathcal{P}^1 u_{2np^j-2} = -u^p_{2np^{j-1}-2}$. From the diagram of homotopy fibrations

$$\begin{array}{ccccc} F_{2t}(n) & \longrightarrow & S^{2np^t-1} & \longrightarrow & J_{p^t-1}\left(S^{2n}\right) \\ \big\downarrow & & \big\downarrow{\scriptstyle E^2} & & \big\| \\ \Omega^2 S^{2n+1} & \xrightarrow{\Omega H_{p^t}} & \Omega^2 S^{2np+1} & \xrightarrow{\partial} & J_{p^t-1}\left(S^{2n}\right) \end{array}$$

we see that $F_{2t}(n)$ is also the homotopy fibre of the composite $S^{2np^t-1} \xrightarrow{E^2} \Omega^2 S^{2np^t+1} \xrightarrow{\partial} J_{p^t-1}\left(S^{2n}\right)$ which will be denoted as f_t. As we shall see, f_t is the attaching map by means of which $J_{p^t}\left(S^{2n}\right)$ is constructed from $J_{p^t-1}\left(S^{2n}\right)$.

Theorem 3.1. *For* $0 \leq q < p^t$, $\Omega J_{p^t+q}\left(S^{2n}\right) \approx F_{2t}(n) \times J\left(\Sigma^{2np^t-1} Y_{q,t}(n)\right)$, *where* $Y_{q,t}(n)$ *is the homotopy pullback*

$$\begin{array}{ccc} Y_{q,t}(n) & \longrightarrow & S^{2np^t-1} \\ \big\downarrow & & \big\downarrow{\scriptstyle f_t} \\ J_q\left(S^{2n}\right) & \hookrightarrow & J_{p^t-1}\left(S^{2n}\right) \end{array} .$$

Proof. From the fibration $J_{p^t-1}\left(S^{2n}\right) \to \Omega S^{2n+1} \to \Omega S^{2np^t+1}$ there is a homotopy action $\tilde{\mu} : \Omega^2 S^{2np^t+1} \times J_{p^t-1}\left(S^{2n}\right) \to J_{p^t-1}\left(S^{2n}\right)$ extending the map $\partial : \Omega^2 S^{2np^t+1} \to J_{p^t-1}\left(S^{2n}\right)$ induced from the fibration. Since $E^2 : S^{2np^t-1} \to \Omega^2 S^{2np^t+1}$ is an H-map, the composite $\mu = \tilde{\mu} \circ \left(E^2 \times J_{p^t-1}\left(S^{2n}\right)\right) : S^{2np^t-1} \times J_{p^t-1}\left(S^{2n}\right) \to J_{p^t-1}\left(S^{2n}\right)$ gives an action of S^{2np^t-1} on $J_{p^t-1}\left(S^{2n}\right)$ which extends f_t. Thus the hypotheses of Theorem 2.1 are satisfied. Applying that theorem with γ equal to the inclusion $J_q\left(S^{2n}\right) \hookrightarrow J_{p^t-1}\left(S^{2n}\right)$ gives $\Omega B \approx F_{2t}(n) \times \Omega\left(S^{2np^t-1} * Y_{k,t}(n)\right)$ where B is the homotopy pushout

$$\begin{array}{ccccc} S^{2np^t-1} \times J_q(S^{2n}) & \longrightarrow & S^{2np^t-1} \times J_{p^t-1}(S^{2n}) & \xrightarrow{\mu} & J_{p^t-1}(S^{2n}) \\ \big\downarrow{\scriptstyle \pi''} & & & & \big\downarrow \\ J_q(S^{2n}) & & \longrightarrow & & B. \end{array}$$

Since $\Omega(M * N) \approx \Omega\Sigma(M \wedge N) \approx J(M \wedge N)$ it suffices to prove the following lemmas which identifies the above pushout as $J_{p^t+q}\left(S^{2n}\right)$.

Lemma 3.2. $B \approx J_{p^t+q}\left(S^{2n}\right)$.

Remark. Applying this lemma in the case $q = 0$ justifies the earlier claim that f_t is the attaching map for the formation of $J_{p^t}\left(S^{2n}\right)$ from $J_{p^t-1}\left(S^{2n}\right)$.

Proof of Lemma 3.2. The homotopy commutative diagram

$$\begin{array}{ccccc}
S^{2np^t-1} \times J_q\left(S^{2n}\right) & \longrightarrow & \Omega^2 S^{2np^t+1} \times J_{p^t-1}\left(S^{2n}\right) & \xrightarrow{\tilde{\mu}} & J_{p^t-1}\left(S^{2n}\right) \\
\downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi''} & & \downarrow \\
J_q\left(S^{2n}\right) & \hookrightarrow & J_{p^t-1}\left(S^{2n}\right) & \longrightarrow & J\left(S^{2n}\right).
\end{array}$$

induces a map $j : B \to J\left(S^{2n}\right)$. Since $J_{p^t+q}\left(S^{2n}\right)$ is the $2n(p^t+q)$-skeleton of $J\left(S^{2n}\right)$, to show that j induces a homology isomorphism (and thus a homotopy equivalence) it suffices to show that $H_*(B) \cong H_*\left(J_{p^t+q}\left(S^{2n}\right)\right)$ and that j induces an injection on homology. From the homotopy cofibration $J_{p^t-1}\left(S^{2n}\right) \to B \to J_q\left(S^{2n}\right)/\left(S^{2np^t-1} \times J_q\left(S^{2n}\right)\right)$ coming from the homotopy pushout defining B we see that

$$\begin{aligned}
H_*(B) &\cong H_*\left(J_{p^t-1}\left(S^{2n}\right)\right) \oplus H_*(S^{2np^t}) \oplus H_*\left(S^{2np^t} \wedge J_q\left(S^{2n}\right)\right) \\
&\cong H_*\left(J_{p^t+q}\left(S^{2n}\right)\right).
\end{aligned}$$

It remains to show that j_* is an injection. From the definitions of B and j we have a diagram of homotopy cofibrations

(1)
$$\begin{array}{ccccc}
S^{2np^t-1} \times J_q\left(S^{2n}\right) & \longrightarrow & J_{p^t-1}\left(S^{2n}\right) & = & J_{p^t-1}\left(S^{2n}\right) \\
\downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\
J_q\left(S^{2n}\right) & \longrightarrow & B & \xrightarrow{j} & J\left(S^{2n}\right) \\
\downarrow & & \downarrow & & \downarrow \\
J_q\left(S^{2n}\right)/\left(S^{2np^t-1} \times J_q\left(S^{2n}\right)\right) & = & B/J_{p^t-1}\left(S^{2n}\right) & \xrightarrow{\tilde{j}} & J\left(S^{2n}\right)/J_{p^t-1}\left(S^{2n}\right).
\end{array}$$

The commutativity of the top right square shows that j_* is an isomorphism in degrees less than or equal to $2n(p^t-1)$. Since we also have the commutative diagram

$$\begin{array}{ccccc}
S^{2np^t-1} \times J_q\left(S^{2n}\right) & \longrightarrow & \Omega^2 S^{2np^t+1} \times J_q\left(S^{2n}\right) & \xrightarrow{\tilde{\mu}} & J_{p^t-1}\left(S^{2n}\right) \\
\downarrow{\scriptstyle \pi''} & & \downarrow{\scriptstyle \pi''} & & \downarrow \\
J_q\left(S^{2n}\right) & \longrightarrow & J_{p^t-1}\left(S^{2n}\right) & \longrightarrow & J\left(S^{2n}\right)
\end{array}$$

the map $\tilde{j}$ in (1) factors as

$$J_q\left(S^{2n}\right)/\left(S^{2np^t-1} \times J_q\left(S^{2n}\right)\right) \xrightarrow{\alpha} J_q\left(S^{2n}\right)/\left(\Omega^2 S^{2np^t+1} \times J_q\left(S^{2n}\right)\right) \xrightarrow{\beta} J\left(S^{2n}\right).$$

The connectivity of α is determined by that of E^2 and in particular α_* is an isomorphism through degree $2np^t-2$. Also the homotopy fibration diagram

$$\begin{array}{ccccc}
\Omega^2 S^{2np^t+1} \times J_{p^t-1}\left(S^{2n}\right) & \xrightarrow{\pi''} & J_{p^t-1}\left(S^{2n}\right) & \xrightarrow{*} & \Omega S^{2np^t+1} \\
\downarrow{\scriptstyle \tilde{\mu}} & & \downarrow & & \| \\
J_{p^t-1}\left(S^{2n}\right) & \longrightarrow & \Omega S^{2n+1} & \xrightarrow{H_{p^t}} & \Omega S^{2np^t+1}
\end{array}$$

gives by Ganea a homotopy fibration diagram

$$\begin{array}{ccccc}
\left(\Omega^2 S^{2np^t+1} \times J_{p^t-1}\left(S^{2n}\right)\right) * \Omega^2 S^{2np^t+1} & \to & \dfrac{J_{p^t-1}\left(S^{2n}\right)}{\Omega^2 S^{2np^t+1} \times J_{p^t-1}\left(S^{2n}\right)} & \to & \Omega S^{2np^t+1} \\
\downarrow & & \downarrow{\scriptstyle\beta} & & \| \\
J_{p^t-1}\left(S^{2n}\right) * \Omega^2 S^{2np^t+1} & \to & \Omega S^{2n+1}/J_{p^t-1}\left(S^{2n}\right) & \to & \Omega S^{2np^t+1}.
\end{array}$$

Since the connectivity of the left map is $2np^t - 1 + 2np^t - 1 > 2n(p^t + q)$ so is that of β. Thus $\tilde{j}_* = \beta\alpha$ is an isomorphism in degrees less than or equal to $2n(p^t+q)$. Therefore the lower right square of (1) shows that j_* is an isomorphism in degrees $2n(p^t-1)+1$ through $2n(p^t+q)$. Combined with our earlier results that j_* is an isomorphism in degrees less than or equal to $2n(p^t-1)$ and $H_*(B) = H_*\left(J_{p^t+q}\left(S^{2n}\right)\right) = 0$ in degrees greater than $2n(p^t+q)$ this shows that j_* is an injection, completing the proof. □

To see that this decomposition is identical to that given in [S4] in the case $t = 1$ we show that $\Sigma Y_{q,1}(n)$ decomposes into the wedge of a sphere and a wedge of Moore spaces. This is a mod p analogue of [S4, Theorem 2.4].

We easily calculate the cohomology of $Y_{q,1}(n)$ to be that given in the following proposition.

Proposition 3.3.

$$H^*\left(Y_{q,1}(n)\right) = \begin{cases} \Lambda(\tilde{a}_{2n(q+1)-1}) \otimes \Gamma(\tilde{u}_{2np-2}) \otimes \Lambda(\tilde{v}_{2np-1}), & \text{if} \quad q < p-1; \\ \Lambda(\tilde{a}_{2np-1}), & \text{if} \quad q = p-1, \end{cases}$$

where $\Lambda(\ )$ *and* $\Gamma(\ )$ *denote respectively the exterior and divided polynomial algebras.* □

Theorem 3.4. *For* $1 \le q < p-1$,

$$\Sigma Y_{q,1}(n) \approx S^{2n(q+1)} \vee \bigvee_{j=1}^{\infty} P^{j(2np-2)+2}(p) \vee \bigvee_{j=1}^{\infty} P^{j(2np-2)+2n(q+1)+1}(p).$$

The corresponding statement when $q = 0$ *holds after one more suspension.*

Proof. We consider first the case $q > 0$. The theorem is equivalent to the statement that the mod p Hurewicz homomorphism

$$\pi_*\left(\Sigma Y_{q,1}(n); \mathbb{Z}/p\mathbb{Z}\right) \to H_*\left(\Sigma Y_{q,1}(n); \mathbb{Z}/p\mathbb{Z}\right)$$

is surjective. From the homotopy pullback defining $Y_{q,1}$ we get a homotopy fibration $F_2(n) \xrightarrow{i} Y_{q,1}(n) \to J_q\left(S^{2n}\right)$. $H_*\big(F_2(n)\big) = \Lambda(b_{2n-1}) \otimes \mathbb{Z}/p\mathbb{Z}(u_{2np-2}) \otimes \Lambda(v_{2np-1})$ and i_* is an isomorphism in degrees $j(2np-2)$ and $j(2np-2)+1$. As in [S2], let K and L denote the $2np-1$-skeletons of $J_{p-1}\left(S^{2n}\right)$ and $F_2(n)$ respectively. From the homotopy pullback defining $Y_{q,1}$ and the H-map $F_2(n) \to S^{2np-1}$ we get a map $\mu : F_2(n) \times Y_{q,1}(n) \to Y_{q,1}(n)$ such that $\mu_*(u^j \otimes 1)$ is non-zero for all j and $\mu_*(1 \otimes a)$ is non-zero, where a is dual to $\tilde{a}$. For $\epsilon = 0$ or 1 and $j > 0$ let $g_{j,\epsilon}$ denote the composite

$$K^{j-1} \times L \times \left(S^{2n(q+1)-1}\right)^{\epsilon} \to F_2(n)^j \times Y_{q,1}(n) \to F_2(n) \times Y_{q,1}(n) \xrightarrow{\mu} Y_{q,1}(n).$$

Then $g_{j,0_*}$ is an isomorphism in degree $j(2np-2)$ and commutativity of μ_* with the coproduct shows that $g_{j,1_*}$ is an isomorphism in degree $j(2np-2)+2n(q+1)-1$. Commutativity with the Bockstein then shows that $g_{j,0_*}$ and $g_{j,1_*}$ are isomorphisms in degrees $j(2np-2)+1$ and $j(2np-2)+2n(q+1)$. The fact that $\Sigma X \times Y \approx \Sigma X \vee \Sigma Y \vee \Sigma X \wedge Y$ together with the splittings $\Sigma^2 L \approx S^{2n+1} \vee P^{2np+1}(p)$ and

$$\Sigma K \wedge L \approx S^{4n-1} \vee S^{2np+2n-2} \vee P^{2np+2n-1}(p) \vee P^{4np-2}(p)$$

shown in [S2] produce maps from Moore spaces which exhibit each element of $H_*\big(\Sigma Y_{q,1}(n)\big)$ above degree $2np$ as an image under the mod p Hurewicz homomorphism. The composite $P^{2np-1}(p) \to F_2(n)/S^{2n-1} \to Y_{q,1}(n)$ together with the map $S^{2n(q+1)-1} \to Y_{q,1}(n)$ in the least non-vanishing degree show the remaining elements are also Hurewicz images.

When $q = 0$, we have $Y_{0,1}(n) = F_2(n)$. The argument is identical to that for $q > 0$ except that there is not a corresponding map $F_2(n)/S^{2n-1} \to Y_{0,1}(n)$ in this case and the elements in degrees $2np-1$ and $2np$ of $\Sigma Y_{0,1}$ are not Hurewicz images. However the decomposition of $\Sigma^2 L$ shows that they become Hurewicz images after one more suspension. $\square$

Corollary 3.5. *For $1 \le q < p-1$, $Y_{q,1}(n) \approx S^{2n(q+1)-1} \times S^{2np-1}\{p\}$.*

Proof. Since $S^{2np-1}\{p\}$ is an H-space, the inclusion $P^{2np-1}(p) \to S^{2np-1}\{p\}$ of the $2np-1$ skeleton extends to a map $\Omega P^{2np}(p) \to S^{2np-1}\{p\}$. Using adjoints of maps coming from the preceding wedge decomposition we get maps $Y_{q,1}(n) \to \Omega P^{2np}(p) \to S^{2np-1}\{p\}$ and $Y_{q,1}(n) \to \Omega S^{2n(q+1)} \to S^{2n(q+1)-1}$ which are onto on homology. Together they give a map $Y_{q,1}(n) \to S^{2n(q+1)-1} \times S^{2np-1}\{p\}$ which induces a homology isomorphism and is thus a homotopy equivalence. $\square$

Corollary 3.6. *For $0 \le q < p$,*

$$\Omega J_{p+q}\left(S^{2n}\right) \approx \begin{cases} F_{2t}(n) \times J\Big(S^{2n(q+1)+2np-2} \vee \\ \qquad \bigvee_{j=1}^{\infty} P^{(j+1)(2np-2)+2}(p) \vee \\ \qquad \bigvee_{j=1}^{\infty} P^{(j+1)(2np-2)+2n(q+1)+1}(p)\Big), & \text{if } q < p; \\ F_{2t}(n) \times J(S^{4np-2}), & \text{if } q = p. \end{cases}$$

This agrees with the result obtained by computational methods in [S4].

Stably there is a Snaith decomposition $\Sigma^\infty \Omega^2 S^{2n+1} \approx \Sigma^\infty \bigvee_{k=0}^{\infty} D_{k,2}(S^{2n-1})$. According to Moore's conjecture (cf. [S3]), suspensions of smash products all of whose factors are $D_{k,2}(S^{2n-1})$ for various k should have exponents for the p-torsion in their homotopy groups. This is known however only when the factors involve k's for which $k < p^2$ so that these spaces are spheres or Moore spaces. Moore's conjecture also claims that $\Omega J_k\left(S^{2n}\right)$ has an exponent for the p-torsion in its homotopy groups for all k. In the case $q = 0$, Theorem 3.1 says $\Omega J_{p^t}\left(S^{2n}\right) \approx F_{2t}(n) \times J\left(\Sigma^{2np-1} F_{2t}(n)\right)$. We will show that $\Sigma^{2np} F_{2t}(n)$ has a wedge decomposition involving suspensions of the spaces $D_{k,2}(S^{2n+1})$ with $k \le p^t$, and their smash products. From this and the Hilton-Milnor theorem it will follow that the problem of showing that Moore's conjecture holds for these suspensions of smash products of spaces occurring in the Snaith decomposition is equivalent to the problem of showing that it holds for all the spaces $\Omega J_{p^t}\left(S^{2n}\right)$. It is not clear at this point which of these problems will be easier.

Lemma 3.7. *If N is large enough so that we have a decomposition $\Sigma^N\Omega^2 S^{2n+1} \approx \Sigma^N \bigvee_{k=0}^{p^t} D_{k,2}(S^{2n-1}) \vee Z$ then*

$$\Sigma^N F_2(n) \approx \Sigma^N \bigvee_{k=0}^{\infty} \left(\big(D_{p^t,2}(S^{2n-1})\big)^{(k)} \wedge \bigvee_{j=0}^{p^t} D_{j,2}(S^{2n-1}) \right),$$

where $X^{(k)}$ denotes the k-fold smash product of X.

Proof. We assign weights to elements in $H_*(F_2(n))$ according to their usual weights in $H_*(\Omega^2 S^{2n+1})$. That is $w(u_{2np^j-2}) = w(v_{2np^j-1}) = p^j$ and $w(xy) = w(x)+w(y)$. For $k \le p^t$ the map $\Sigma^N \bigvee_{j=0}^{p^t} D_{j,2}(S^{2n-1}) \xrightarrow{\alpha} \Sigma^N\Omega^2 S^{2n+1}$ lifts for connectivity reasons to $\Sigma^N F_2(n)$. Let $\alpha_j : D_{j,2}(S^{2n-1}) \to \Sigma^N F_2(n)$ be the restriction of α to the j th piece. Let g_k be the composite

$$\Sigma^N \big(D_{p^t,2}(S^{2n-1})\big)^{(k)} \wedge \bigvee_{j=0}^{p^t} D_{j,2}(S^{2n-1}) \xrightarrow{\alpha_{p^t}^{(k)} \wedge \bigvee_{j=0}^{p^t} \alpha_j} \Sigma^N \big(F_2(n)\big)^{(k+1)}$$

$$\xrightarrow{s_{k+1}} \Sigma^N \big(F_2(n)\big)^{k+1} \to \Sigma^N F_2(n)$$

where some homotopy splitting $s_m : \Sigma^N \big(F_2(n)\big)^{(m)} \to \Sigma^N \big(F_2(n)\big)^m$ to the canonical map in the other direction has been chosen. The map $\bigvee_{k=0}^{\infty} g_k$ is a homology isomorphism and thus a homotopy equivalence. □

According to [C], to satisfy the hypotheses of Lemma 3.7 it is sufficient for N to be at least $2p^t$ and thus in particular we get the decomposition for $\Sigma^{2np^t} F_{2t}(n)$. Thus we have the following corollary.

Corollary 3.8.

$$\Omega J_{p^t}\left(S^{2n}\right) \approx F_{2t}(n) \times J\left(\Sigma^{2np^t-1} \bigvee_{k=0}^{\infty} \left(\big(D_{p^t,2}(S^{2n-1})\big)^{(k)} \wedge \bigvee_{j=0}^{p^t} D_{j,2}(S^{2n-1}) \right)\right).$$

□

It is clear from the definition of $F_{2t}(n)$ that the p-torsion of its homotopy groups has an exponent. Therefore if all spaces formed as suspensions of smash products of spaces occurring in the Snaith decomposition of $\Omega^2 S^{2n+1}$ have mod p homotopy exponents then $J_{p^t}\left(S^{2n}\right)$ has a mod p homotopy exponent for all t. Conversely a mod p homotopy exponent for $J_{p^t}\left(S^{2n}\right)$ implies a mod p homotopy exponent for a large number of such suspensions of smash products.

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Department of Mathematics, University of Toronto, 100 St. George St., Toronto, Ontario, Canada M5S 1A1

E-mail address: `selick@math.toronto.edu`